

On the Fučík spectrum of the wave operator and an asymptotically linear problem*

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Abstract

We study generalized solutions of the nonlinear wave equation

$$u_{tt} - u_{ss} = au^+ - bu^- + p(s, t, u),$$

with periodic conditions in t and homogeneous Dirichlet conditions in s , under the assumption that the ratio of the period to the length of the interval is two. When $p \equiv 0$ and λ is a nonzero eigenvalue of the wave operator, we give a proof of the existence of two families of curves (which may coincide) in the Fučík spectrum intersecting at (λ, λ) . This result is known for some classes of self-adjoint operators (which does not cover the situation we consider here), but in a smaller region than ours. Our approach is based on a dual variational formulation and is also applicable to other operators, such as the Laplacian. In addition, we prove an existence result for the nonhomogeneous situation, when the pair (a, b) is not ‘between’ the Fučík curves passing through $(\lambda, \lambda) \neq (0, 0)$ and p is a continuous function, sublinear at infinity.

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1 Introduction

Let Ω be an open bounded region in \mathbb{R}^N and $L : D(L) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ a self-adjoint operator. Consider the equation

$$Lu = \alpha u^+ - \beta u^-, \quad u \in D(L)$$

where $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$. The set of points (α, β) for which the equation has a nonzero solution is called the Fučík spectrum of L . Since the pioneering work of Fučík [9], there has been a growing interest in the study of the structure of this set. In particular, if λ is an eigenvalue of L , then clearly (λ, λ) is on the Fučík spectrum. Of special interest is the study of connected components of the Fučík spectrum that meet at (λ, λ) . There are a number of works concerning the structure of the Fučík spectrum for operators such as the Laplace operator or, more generally, a self-adjoint operator with a compact resolvent. But if one turns to the wave operator or to operators with noncompact resolvent the list becomes considerably smaller.

Here we only mention the paper [3]. In this work Ben-Naoum, Fabry and Smets apply a Lyapunov-Schmidt decomposition together with some contraction mapping arguments to give a description of the Fučík spectrum of an operator away from its essential spectrum, provided that some non-degeneracy conditions are satisfied. These conditions do not seem to hold for the wave operator however.

Before describing the main ideas in our approach, we briefly recall some of the work on the Fučík spectrum of a self-adjoint operator with compact resolvent. In the one dimensional case the Fučík spectrum was completely described by Fučík in [9]. Some of our spectrum will arise from this one dimensional case (see Section 2). If the space dimension is greater than one, the complete understanding of the Fučík spectrum has been more difficult. Ambrosetti and Prodi [2] obtained two nonincreasing curves when the nonlinearity crosses the first eigenvalue. This result was extended by Gallouët and Kavian [10] and Ruf [21] to the case where the nonlinearity crosses a higher eigenvalue, in the case that the eigenvalue is simple. Căc [4] generalized the result for the Laplacian operator with Dirichlet boundary conditions. Abchir [1] extended the result to a self-adjoint operator with compact resolvent whose spectrum is not bounded below. If λ_k is the k -th eigenvalue of L , we note that the existence of the curves passing through the point (λ_k, λ_k) , is usually established in the square $\lambda_{k-1} \leq a, b \leq \lambda_{k+1}$. We also mention that Marino, Micheletti and Pistoia in [12] proved a detailed result regarding some curves belonging to the Fučík spectrum of the Laplacian on a bounded domain with Dirichlet conditions. See also [19] where a different characterization of the curves is given.

Finally in [8], de Figueiredo and Gossez prove the existence of a first Fučík curve through (λ_2, λ_2) which extends to infinity. Their result is for a general elliptic operator in divergence form, and they provide a variational characterization of the curve. In fact, drawing a line of positive slope from (λ_1, λ_1) , de Figueiredo and Gossez obtain the first intersection point with the Fučík spectrum through some constrained minimization of a Dirichlet type integral. This work provided the initial inspiration for our paper. Indeed, applying a similar variational characterization, but to a suitably *shifted* dual problem, we are able to prove the existence of two continuous curves through any nonzero eigenvalue point (λ_k, λ_k) of the wave operator. These curves, in a sense made precise below, are the extreme parts of the Fučík spectrum inside a rectangle containing the eigenvalue point. It is worth mentioning that our approach can not only handle other operators with noncompact resolvent (such as the beam operator), but can also be applied to classes of operators that have already been studied in the works mentioned above, providing new proofs for known results but in regions that are larger than the square $[\lambda_{k-1}, \lambda_{k+1}]^2$.

In [6] Choi, McKenna and Romano also used a dual variational formulation and the mountain pass lemma to prove existence of multiple periodic solutions of the semilinear vibrating string model problem.

After completion of this work we learned that Nečesal [20] had proposed a somewhat similar approach (with no theoretical justification) to obtain a numerical algorithm to explore parts of the Fučík spectrum of the wave and beam operators. Since the kernel of the wave operator is infinite dimensional, it is not a priori clear that the dual problem has a solution. In fact, in order to prove convergence of the maximizing sequences, we have to employ an appropriate second shift which is absent in Nečesal's description.

Fučík [9] and Dancer [7] were the first to recognize the importance of the Fučík spectrum in the study of semilinear boundary value problems with linear growth at infinity. Theorems on existence of solutions of the non-homogeneous equation either treat so called type-I regions (when the pair (a, b) does not lie between the two curves above through a (λ_k, λ_k)), or treat so called type-II regions (when the pair (a, b) lies between the two curves above through a (λ_k, λ_k)). There is a substantial amount of work done for type-I regions in the case of a self-adjoint operator with compact resolvent. Results for type-II regions are proved in the paper [3], where more general self-adjoint operators are also considered, as mentioned above.

Several papers study the non-homogeneous wave equation. Using the same boundary conditions that we use, in [15] McKenna considers the case where $a = b$, both at resonance and nonresonance, by reduction to a Landesman-Lazer problem. In [24] Willem, with periodicity conditions on both

variables, overcomes the fact that the kernel of the wave operator is infinite dimensional by proving a Continuation Theorem. Under appropriate conditions, he proves the existence of at least one generalized periodic solution of

$$u_{tt} - u_{ss} = au^+ - bu^- + p,$$

with $\lim_{|u| \rightarrow +\infty} p(s, t, u)/u = 0$, but only for a, b in a box $0 < \mu \leq a, b \leq \nu$, $(\mu, \mu) \neq (a, b) \neq (\nu, \nu)$. Here μ, ν are two consecutive elements of the spectrum of the wave operator. In [17] McKenna, Redlinger and Walter prove existence and multiplicity results for an asymptotically homogeneous hyperbolic problem when the nonlinearity is monotone in u . They are particularly interested in the situation when the nonlinearity crosses several eigenvalues. They reduce the problem to one on a subspace on which the linear operator has a compact inverse; then they apply degree theory. In [11] Lazer and McKenna obtain at least two solutions of a wave equation when the nonlinearity crosses the first eigenvalue and is monotone increasing. In our work, using our dual variational approach, we will also consider the nonhomogeneous equation. We prove existence of a weak solution for the parameters a, b in larger type-I regions of the plane, under a geometric condition for the linearization with respect to u of the nonlinearity.

We remark that if one restricts to a space of solutions with certain symmetries, then one can remove zero from the spectrum and end up with a compact operator, avoiding the difficulties one usually finds in this type of problems. This idea is due independently to Coron [5] and Vejvoda [22]. It has been used by McKenna in [18] to investigate nonlinear oscillations in a suspension bridge.

Finally, in [16] McKenna obtains results for some situations in which the ratio of the period to the length of the interval is irrational, sometimes referred to as a small divisors problem. Two important differences arise. The wave operator is invertible, and each point of the spectrum is an eigenvalue of infinite multiplicity.

The organization of this work is as follows. First, in Sections 2 and 3 we treat the positive-homogeneous equation. Indeed, in Section 2 we give a dual variational formulation for the Fučík curves, and in Section 3 we prove existence of solutions of the dual problems. In Section 4 we present and prove an existence result for the non-homogeneous equation. Our main results are Theorems 3.4, 3.5, 4.4 and 4.5.

2 Two equivalent problems

We denote by \mathbb{T} the circle $\mathbb{R}/(2\pi\mathbb{Z})$. Consider the wave operator defined on $\{u \in H^2([0, \pi] \times \mathbb{T}) : u(0, t) = u(\pi, t) = 0 \text{ for } t \in \mathbb{T}\}$. Its eigenvalues are given by $\lambda_{(m,n)} = m^2 - n^2$ for any $(m, n) \in \mathbb{N} \times \mathbb{N}_0$. The functions

$$\phi_{(m,n)} = \frac{\sqrt{2}}{\pi} \sin(ms) \cos(nt), \quad \psi_{(m,n)} = \frac{\sqrt{2}}{\pi} \sin(ms) \sin(nt)$$

are eigenfunctions associated to $\lambda_{(m,n)}$ (see [13]). We let $\mathcal{H} = L^2([0, \pi] \times \mathbb{T})$,

$$\mathcal{H} = \left\{ v = \sum [\alpha_{(m,n)} \phi_{(m,n)} + \beta_{(m,n)} \psi_{(m,n)}] : \sum [\alpha_{(m,n)}^2 + \beta_{(m,n)}^2] < \infty \right\}, \quad (1)$$

and define

$$\mathcal{R} = \overline{\text{span}\{\phi_{(m,n)}, \psi_{(m,n)} \text{ with } m \in \mathbb{N}, n \in \mathbb{N}_0, m \neq n\}},$$

$$\mathcal{N} = \overline{\text{span}\{\phi_{(l,l)}, \psi_{(l,l)} \text{ with } l \in \mathbb{N}\}},$$

where the closures are in \mathcal{H} , so that $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$. Let also $\mathcal{H}^1 = \mathcal{R} \cap H^1([0, \pi] \times \mathbb{T})$. We denote by $\dots \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$ the eigenvalues ordered according to their value. Suppose that $\lambda_k \neq 0$. We are concerned with finding continuous functions a and b from \mathbb{R}^+ to \mathbb{R} , satisfying $a(1) = b(1) = \lambda_k$, such that for each $r \in \mathbb{R}^+$ the problem

$$\square u = u_{tt} - u_{ss} = a(r)u^+ - b(r)u^- \quad \text{in }]0, \pi[\times \mathbb{T}, \quad (2)$$

$$u(0, t) = u(\pi, t) = 0 \quad \text{for } 0 \leq t \leq 2\pi, \quad (3)$$

has a nonzero weak solution u . As before $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$. By a weak solution we mean a critical point of the energy functional \mathcal{I} , defined on $\mathcal{H}^1 \times \mathcal{N}$;

$$\mathcal{I}(x, y) = \frac{1}{2} (\|x_s\|^2 - \|x_t\|^2 - a\|(x+y)^+\|^2 - b\|(x+y)^-\|^2).$$

A word on notation. Throughout $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathcal{H} and $\|\cdot\|$ the corresponding norm.

We identify the pair (x, y) with $u = x + y$. Since the functions $\phi_{(m,n)}$ and $\psi_{(m,n)}$ satisfy (3), the trace theorem implies that any $x \in \mathcal{H}^1$ has L^2 trace on $\{0\} \times \mathbb{T} \cup \{\pi\} \times \mathbb{T}$ satisfying (3). On the other hand, any function $y \in \mathcal{N}$ is of the form $y(s, t) = \kappa(t + s) - \kappa(t - s)$ for some $\kappa \in L^2(\mathbb{T})$ and thus also satisfies (3). Therefore, any critical point of \mathcal{I} satisfies the boundary conditions (3).

We should mention that some of the Fučík spectrum will come from solutions of the form

$$u(s, t) = \sin s \times T(t)$$

with $T'' + T = aT^+ - bT^-$, and from solutions of the form

$$u(s, t) = S(s) \times 1$$

with $-S'' = aS^+ - bS^-$. This produces Fučík curves through the eigenvalues $1 - n^2$ and m^2 , respectively. The curves can be obtained explicitly (see [9]).

2.1 Variational formulation for the Fučík curve through (λ_k, λ_k) closest to $(\lambda_{k-1}, \lambda_{k-1})$

Let k be a nonzero integer. Choose a small parameter ε_1 with $0 < \varepsilon_1 < \lambda_k - \lambda_{k-1}$. Suppose u is a weak solution of (2)-(3). Then u solves

$$(\square - (\lambda_{k-1} + \varepsilon_1))u = (a - \lambda_{k-1} - \varepsilon_1)u^+ - (b - \lambda_{k-1} - \varepsilon_1)u^- \quad (4)$$

together with (3). We write u as

$$u = \sum [\theta_{(m,n)}\phi_{(m,n)} + \omega_{(m,n)}\psi_{(m,n)}]. \quad (5)$$

Suppose $a, b > \lambda_{k-1} + \varepsilon_1$. Consider the convex function $H : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$H(u) = \frac{1}{2}(a - \lambda_{k-1} - \varepsilon_1)(u^+)^2 + \frac{1}{2}(b - \lambda_{k-1} - \varepsilon_1)(u^-)^2. \quad (6)$$

Let

$$v := \nabla H(u) = (a - \lambda_{k-1} - \varepsilon_1)u^+ - (b - \lambda_{k-1} - \varepsilon_1)u^-.$$

From (4) we deduce that $u = Lv$ for $L : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$Lv = \sum \left[\frac{\alpha_{(m,n)}}{m^2 - n^2 - \lambda_{k-1} - \varepsilon_1} \phi_{(m,n)} + \frac{\beta_{(m,n)}}{m^2 - n^2 - \lambda_{k-1} - \varepsilon_1} \psi_{(m,n)} \right]$$

and v written as in (1). Hence the function v satisfies

$$Lv = \nabla H^*(v) = \frac{1}{a - \lambda_{k-1} - \varepsilon_1}v^+ - \frac{1}{b - \lambda_{k-1} - \varepsilon_1}v^-. \quad (7)$$

where H^* is the Fenchel-Legendre transform of the convex function H .

Conversely, suppose now $v \in \mathcal{H}$ is a solution of (7). We may decompose $Lv = x + y$ with $x \in \mathcal{R}$ and $y \in \mathcal{N}$. In fact, the next lemma shows $x \in \mathcal{H}^1$.

Lemma 2.1. *Let v belong to \mathcal{H} and x be the component of Lv in \mathcal{R} . Then x belongs to the space \mathcal{H}^1 .*

Proof. The function x may be expanded as

$$x = \sum_{|m-n| \geq 1} \left[\frac{\alpha_{(m,n)}}{m^2 - n^2 - \lambda_{k-1} - \varepsilon_1} \phi_{(m,n)} + \frac{\beta_{(m,n)}}{m^2 - n^2 - \lambda_{k-1} - \varepsilon_1} \psi_{(m,n)} \right].$$

Since $\lambda_{k-1} + \varepsilon_1$ is not an eigenvalue of the wave operator, there exists $\delta > 0$ such that

$$|m^2 - n^2 - \lambda_{k-1} - \varepsilon_1| > \delta \quad (8)$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Without loss of generality, we assume that $\lambda_{k-1} + \varepsilon_1 > 0$. Otherwise interchange the roles of m and n below. We claim that there exists $\varepsilon > 0$ such that $|m - \sqrt{n^2 + \lambda_{k-1} + \varepsilon_1}| > \varepsilon$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ with $|m - n| \geq 1$. We prove this assertion by contradiction. Suppose there exist a sequence (m_l, n_l) with

$$|m_l - n_l| \geq 1 \quad (9)$$

and $|m_l - \sqrt{n_l^2 + \lambda_{k-1} + \varepsilon_1}| \leq 1/l$. Either m_l and n_l are both bounded, or both sequences are unbounded. In the former case, modulo a subsequence, $m_l = m_0$ and $n_l = n_0$ for large l and $m_0 = \sqrt{n_0^2 + \lambda_{k-1} + \varepsilon_1}$. This contradicts inequality (8). In the latter case,

$$\begin{aligned} |m_l - n_l| &\leq \left| m_l - \sqrt{n_l^2 + \lambda_{k-1} + \varepsilon_1} \right| + \left| \sqrt{n_l^2 + \lambda_{k-1} + \varepsilon_1} - n_l \right| \\ &\leq \frac{1}{l} + \frac{\lambda_{k-1} + \varepsilon_1}{\sqrt{n_l^2 + \lambda_{k-1} + \varepsilon_1} + n_l} \rightarrow 0 \quad \text{as } l \rightarrow \infty, \end{aligned}$$

which contradicts (9). This completes the proof of the claim. Hence

$$\frac{m^2 + n^2}{(m^2 - n^2 - \lambda_{k-1} - \varepsilon_1)^2} \leq \frac{1}{\varepsilon^2} \frac{m^2 + n^2}{(m + \sqrt{n^2 + \lambda_{k-1} + \varepsilon_1})^2} \leq \frac{1}{\varepsilon^2}$$

implying that $x \in \mathcal{H}^1$. □

Let $u = Lv = \nabla H^*(v)$. If we write v as in (1) and u as in (5), then $(m^2 - n^2 - \lambda_{k-1} - \varepsilon_1)\theta_{(m,n)} = \alpha_{(m,n)}$ and $(m^2 - n^2 - \lambda_{k-1} - \varepsilon_1)\omega_{(m,n)} = \beta_{(m,n)}$. Therefore u is a weak solution of (4) together with (3).

We now choose a small parameter ε_2 with

$$0 < \varepsilon_2 < \frac{1}{\lambda_k - \lambda_{k-1} - \varepsilon_1} - \max \left\{ 0, -\frac{1}{\lambda_{k-1} + \varepsilon_1} \right\}$$

and take

$$\mu = \max \left\{ 0, -\frac{1}{\lambda_{k-1} + \varepsilon_1} \right\} + \varepsilon_2. \quad (10)$$

We rewrite (7) as

$$(L - \mu)v = \hat{a}v^+ - \hat{b}v^-, \quad (11)$$

with

$$\hat{a} = \frac{1}{a - \lambda_{k-1} - \varepsilon_1} - \mu, \quad \hat{b} = \frac{1}{b - \lambda_{k-1} - \varepsilon_1} - \mu. \quad (12)$$

Note that the greatest eigenvalue of $L - \mu$ is $1/(\lambda_k - \lambda_{k-1} - \varepsilon_1) - \mu > 0$ and the space \mathcal{N} is associated to the eigenvalue $\nu := -1/(\lambda_{k-1} + \varepsilon_1) - \mu < 0$.

Fixing $r > 0$, we consider solutions of (11) on the line $\hat{b} = r\hat{a}$:

$$(L - \mu)v = \hat{a}(v^+ - rv^-). \quad (13)$$

Next we introduce the C^1 functionals F and G from \mathcal{H} to \mathbb{R} by

$$\begin{aligned} F(v) &= \langle (L - \mu)v, v \rangle, \\ G(v) &= \|v^+\|^2 + r\|v^-\|^2. \end{aligned}$$

To find the point on the line $\hat{b} = r\hat{a}$ farthest away from the origin for which (13) has a nonzero solution we consider the maximization problem

$$\sup_{G(v)=1} F(v) = \sup_{v \neq 0} \frac{F(v)}{G(v)}, \quad v \in \mathcal{H}. \quad (14)$$

Finally $\check{a}(r)$ denotes the value of this supremum.

2.2 Variational formulation for the Fučík curve through (λ_k, λ_k) closest to $(\lambda_{k+1}, \lambda_{k+1})$

Choose a small parameter ε_3 with $0 < \varepsilon_3 < \lambda_{k+1} - \lambda_k$. If u is a weak solution of (2)-(3) then u solves

$$(-\square + \lambda_{k+1} - \varepsilon_3)u = (\lambda_{k+1} - a - \varepsilon_3)u^+ - (\lambda_{k+1} - b - \varepsilon_3)u^- \quad (15)$$

together with (3). Suppose $a, b < \lambda_{k+1} - \varepsilon_3$. Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be the convex function defined by

$$K(u) = \frac{1}{2}(\lambda_{k+1} - a - \varepsilon_3)(u^+)^2 + \frac{1}{2}(\lambda_{k+1} - b - \varepsilon_3)(u^-)^2,$$

and $M : \mathcal{H} \rightarrow \mathcal{H}$ be defined by

$$Mv = \sum \left[\frac{\alpha_{(m,n)}}{\lambda_{k+1} - m^2 + n^2 - \varepsilon_3} \phi_{(m,n)} + \frac{\beta_{(m,n)}}{\lambda_{k+1} - m^2 + n^2 - \varepsilon_3} \psi_{(m,n)} \right],$$

for v as in (1). If $v = \nabla K(u)$, then

$$Mv = \nabla K^*(v) = \frac{1}{\lambda_{k+1} - a - \varepsilon_3} v^+ - \frac{1}{\lambda_{k+1} - b - \varepsilon_3} v^-. \quad (16)$$

And conversely, any solution of (16) is a weak solution of (15) together with (3). Finally, we choose a small parameter ε_4 with

$$0 < \varepsilon_4 < \frac{1}{\lambda_{k+1} - \lambda_k - \varepsilon_3} - \max \left\{ 0, \frac{1}{\lambda_{k+1} - \varepsilon_3} \right\}$$

and take

$$\rho = \max \left\{ 0, \frac{1}{\lambda_{k+1} - \varepsilon_3} \right\} + \varepsilon_4.$$

We rewrite (16) as

$$(M - \rho)v = \bar{a}v^+ - \bar{b}v^-, \quad (17)$$

with

$$\bar{a} = \frac{1}{\lambda_{k+1} - a - \varepsilon_3} - \rho, \quad \bar{b} = \frac{1}{\lambda_{k+1} - b - \varepsilon_3} - \rho.$$

Note that the greatest eigenvalue of $M - \rho$ is $1/(\lambda_{k+1} - \lambda_k - \varepsilon_3) - \rho > 0$ and the space \mathcal{N} is associated to the eigenvalue $\sigma := 1/(\lambda_{k+1} - \varepsilon_3) - \rho < 0$.

Again fixing $r > 0$, we consider solutions of (17) on the line $\bar{b} = r\bar{a}$:

$$(M - \rho)v = \bar{a}(v^+ - rv^-). \quad (18)$$

Finally we consider

$$\sup_{v \neq 0} \frac{\langle (M - \rho)v, v \rangle}{G(v)}, \quad v \in \mathcal{H} \quad (19)$$

and denote by $\tilde{a}(r)$ the value of this supremum.

3 Proof of existence of solutions of the dual problems

In this section we prove the existence of maximizers for the problems (14) and (19) and examine simple properties of the maxima $\check{a}(r)$ and $\tilde{a}(r)$. Translating these results in terms of the parameters of the original equation we obtain the two curves in the Fučík spectrum stated in Theorems 3.4 and 3.5.

Proposition 3.1. *The supremum in (14) is attained.*

Proof. Let v_n be a maximizing sequence for (14) such that $G(v_n) = 1$. We write $v_n = w_n + z_n$ with $w_n \in \mathcal{R}$ and $z_n \in \mathcal{N}$. Modulo a subsequence, we may assume that $v_n^+ \rightharpoonup \gamma$, $v_n^- \rightharpoonup \eta$, $v_n \rightharpoonup v_0$, $w_n \rightharpoonup w_0$ and $z_n \rightharpoonup z_0$, all in \mathcal{H} . We remark that $v_0 = \gamma - \eta = v_0^+ - v_0^-$ with $\gamma \geq v_0^+$ and $\eta \geq v_0^-$. We prove that $\limsup \|w_n\| = \|w\|$ and $\limsup \|z_n\| = \|z\|$. This will imply that $\lim \|w_n\| = \|w_0\|$ and $\lim \|z_n\| = \|z_0\|$, so that $v_n \rightarrow v$ in \mathcal{H} . To do so we prove that for any subsequence along which both $\|w_n\|$ and $\|z_n\|$ converge, we have $\lim \|w_n\| = \|w_0\|$ and $\lim \|z_n\| = \|z_0\|$. So suppose that $\|w_n\|$ and $\|z_n\|$ converge. Since $Lw_n \rightarrow Lw_0$ strongly in \mathcal{H} (see Lemma 2.1), we have

$$\begin{aligned} \check{a}(r) &= \lim \langle (L - \mu)v_n, v_n \rangle \\ &= \lim [\langle (L - \mu)w_n, w_n \rangle + \langle (L - \mu)z_n, z_n \rangle] \\ &= \lim [\langle (L - \mu)w_0, w_0 \rangle - \mu(\|w_n\|^2 - \|w_0\|^2) \\ &\quad + \langle (L - \mu)z_0, z_0 \rangle + \nu(\|z_n\|^2 - \|z_0\|^2)] \\ &= \langle (L - \mu)v_0, v_0 \rangle - \mu \lim(\|w_n\|^2 - \|w_0\|^2) + \nu \lim(\|z_n\|^2 - \|z_0\|^2). \end{aligned}$$

Recalling that $\mu > 0$ and $\nu < 0$, we conclude

$$F(v_0) = \langle (L - \mu)v_0, v_0 \rangle \geq \check{a}(r), \quad (20)$$

and the inequality is strict, unless $\lim \|w_n\| = \|w_0\|$ and $\lim \|z_n\| = \|z_0\|$. On the other hand from

$$G(v_n) = \|v_n^+\|^2 + r\|v_n^-\|^2 = 1$$

it follows that

$$G(v_0) = \|v_0^+\|^2 + r\|v_0^-\|^2 \leq \|\gamma\|^2 + r\|\eta\|^2 \leq 1. \quad (21)$$

We know that $\check{a}(r) > 0$ by testing F/G with an eigenfunction associated to λ_k . Thus inequality (20) shows that $v_0 \neq 0$. Clearly, $F(v_0)/G(v_0) \leq \check{a}(r)$, so (20) and (21) combined imply that $G(v_0) = 1$, $F(v_0) = \check{a}(r)$ and the sequence v_n converges to v_0 strongly in \mathcal{H} . \square

Remark 3.2. *The purpose of the second shift $-\mu I$ is twofold. First, to guarantee that the eigenvalue ν of $L - \mu$ associated to \mathcal{N} is negative. And second, given a maximizing sequence, to guarantee the convergence of the sequence of components in the range of the wave operator.*

We may write

$$\check{a}(r) = F(v_0)/G(v_0). \quad (22)$$

For all $\varphi \in \mathcal{H}$ we have

$$\left(\frac{F}{G} \right)' \Big|_{v=v_0} \varphi = \frac{F'(v_0) - \check{a}(r)G'(v_0)}{G(v_0)} \varphi = 0,$$

so that

$$F'(v_0) = \check{a}(r)G'(v_0).$$

We conclude that v_0 is a nonzero solution of (13) with $\hat{a} = \check{a}(r)$. And there can be no nontrivial solution of (13) with $\hat{a} > \check{a}(r)$ for otherwise the functional F/G would have a critical value greater than its maximum.

Observe that the function \check{a} is strictly decreasing unless the maximizer v_0 has a fixed sign. If so, the function v_0 is an eigenfunction and it must correspond to $\lambda_{(1,0)} = 1$.

Lemma 3.3. *The function \check{a} is continuous.*

Proof. Let $v_0(r)$ be such that $\check{a}(r) = F(v_0(r))$ with $G_r(v_0(r)) = 1$. Let $r_0 > 0$. The inequality

$$\check{a}(r_0) \geq \frac{F(v_0(r))}{G_{r_0}(v_0(r))} = \frac{\check{a}(r)}{G_r(v_0(r)) + (r_0 - r)\|v_0^-(r)\|^2}$$

implies

$$\limsup_{r \rightarrow r_0} \check{a}(r) \leq \check{a}(r_0).$$

On the other hand, the inequality

$$\check{a}(r) \geq \frac{F(v_0(r_0))}{G_r(v_0(r_0))} = \frac{\check{a}(r_0)}{G_{r_0}(v_0(r_0)) + (r - r_0)\|v_0^-(r_0)\|^2}$$

implies

$$\liminf_{r \rightarrow r_0} \check{a}(r) \geq \check{a}(r_0).$$

This proves the continuity of \check{a} at r_0 . □

Let us now explicitly indicate the dependence of G on r by writing G_r . The equality

$$\frac{F(-v)}{G_{\frac{1}{r}}(-v)} = r \frac{F(v)}{G_r(v)},$$

valid for all $v \neq 0$, implies that

$$\check{a}\left(\frac{1}{r}\right) = r\check{a}(r). \tag{23}$$

So

$$\left(\check{a}\left(\frac{1}{r}\right), \frac{1}{r}\check{a}\left(\frac{1}{r}\right)\right) = (r\check{a}(r), \check{a}(r)). \tag{24}$$

We can now state

Theorem 3.4. Fix $\lambda_k \neq 0$. If $\lambda_k > 0$ let $\check{Q}_k =]\lambda_{k-1}, +\infty[^2$, and if $\lambda_k < 0$ let $\check{Q}_k =]\lambda_{k-1}, 0]^2$. There exists a continuous curve $\mathcal{C}_k \subset \check{Q}_k$ through (λ_k, λ_k) (symmetric with respect to the line that bisects the odd quadrants and, if $k \neq 1$, the graph of a strictly decreasing function) such that $(a, b) \in \mathcal{C}_k$ implies (2)-(3) has a nontrivial weak solution. If (a, b) lies in \check{Q}_k and below \mathcal{C}_k , then (2)-(3) has no nontrivial weak solution.

Proof. Let $0 < r < \infty$ be fixed and consider the half-line $\hat{b} = r\hat{a}$ with $\hat{a} > 0$ in the (\hat{a}, \hat{b}) -plane. We have seen that there are no nontrivial solutions of (11) on this line with $\hat{a} > \check{a}(r)$. We can write the original parameters a and b in (2) in terms of \hat{a} and \hat{b} as

$$a = \lambda_{k-1} + \varepsilon_1 + \frac{1}{\hat{a} + \mu}, \quad b = \lambda_{k-1} + \varepsilon_1 + \frac{1}{\hat{b} + \mu}. \quad (25)$$

As we increase \hat{a} from zero to infinity, the image of the half-line in the (a, b) -plane is a curve starting at $(\lambda_{k-1} + \varepsilon_1 + 1/\mu, \lambda_{k-1} + \varepsilon_1 + 1/\mu)$ and ending at $(\lambda_{k-1} + \varepsilon_1, \lambda_{k-1} + \varepsilon_1)$. The map given by (25) is a bijection between $]0, +\infty[^2$ in the (\hat{a}, \hat{b}) plane and

$$Q_k :=]p_k, q_k[^2 :=]\lambda_{k-1} + \varepsilon_1, \lambda_{k-1} + \varepsilon_1 + 1/\mu[^2 \quad (26)$$

in the (a, b) plane. If $\lambda_k > 0$ then $Q_k =]\lambda_{k-1} + \varepsilon_1, \lambda_{k-1} + \varepsilon_1 + 1/\varepsilon_2[^2$, and if $\lambda_k < 0$ then

$$Q_k = \left] \lambda_{k-1} + \varepsilon_1, -\varepsilon_2 \frac{(\lambda_{k-1} + \varepsilon_1)^2}{1 - \varepsilon_2(\lambda_{k-1} + \varepsilon_1)} \right]^2.$$

Consider the curve \mathcal{C}_k parametrized by

$$r \mapsto \left(\lambda_{k-1} + \varepsilon_1 + \frac{1}{\check{a}(r) + \mu}, \lambda_{k-1} + \varepsilon_1 + \frac{1}{r\check{a}(r) + \mu} \right) := (a(r), b(r)).$$

Clearly, $\check{a}(1)$ is the greatest eigenvalue of $L - \mu$, so the curve \mathcal{C}_k passes through (λ_k, λ_k) . As $r \rightarrow +\infty$, either $b(r) \rightarrow \lambda_{k-1} + \varepsilon_1$ (if $\lim_{r \rightarrow +\infty} \check{a}(r) > 0$), or $a(r) \rightarrow \lambda_{k-1} + \varepsilon_1 + 1/\mu$ (if $\lim_{r \rightarrow +\infty} \check{a}(r) = 0$), or both. On the other end, as $r \rightarrow 0$, either $a(r) \rightarrow \lambda_{k-1} + \varepsilon_1$ (if $\lim_{r \rightarrow 0} \check{a}(r) = +\infty$), or $b(r) \rightarrow \lambda_{k-1} + \varepsilon_1 + 1/\mu$ (if $\lim_{r \rightarrow 0} \check{a}(r) < +\infty$), or both. Therefore \mathcal{C}_k approaches the boundary of Q_k as $r \rightarrow 0$ and $r \rightarrow +\infty$.

From (23), note also that $(\check{a}(r), r\check{a}(r)) = (\check{a}(r), \check{a}(\frac{1}{r}))$. Suppose $\lambda_k \neq 1$. As r increases, $\check{a}(r)$ decreases and $\check{a}(\frac{1}{r})$ increases. The curve \mathcal{C}_k is the graph of a strictly decreasing function $b = b(a)$. It lies in $\{(a, b) \in \mathbb{R}^2 : b < \lambda_k < a \text{ or } a < \lambda_k < b \text{ or } a = \lambda_k = b\}$. In addition (24) implies that the curve \mathcal{C}_k is symmetric with respect to the line $b = a$.

Lemma 3.3 implies that \mathcal{C}_k is continuous. The curve \mathcal{C}_k divides the square Q_k into two connected components. We say that a point is inside Q_k and *below* \mathcal{C}_k if it belongs to the component which contains $]\lambda_{k-1} + \varepsilon_1, \lambda_k[^2$. There are no points on the Fučík spectrum of (2)-(3) inside the square Q_k and below \mathcal{C}_k .

Finally we explicitly indicate the dependence of the square Q_k on ε_1 and ε_2 by writing $Q_k = Q_k(\varepsilon_1, \varepsilon_2)$. If $\lambda_k > 0$ we define $\check{Q}_k :=]\lambda_{k-1}, +\infty[^2$, and if $\lambda_k < 0$ we define $\check{Q}_k :=]\lambda_{k-1}, 0[^2$. If $(a, b) \in \check{Q}_k$ then we can choose $\varepsilon_1, \varepsilon_2$ sufficiently small so that $(a, b) \in Q_k(\varepsilon_1, \varepsilon_2)$.

Note that if we consider two squares Q_k , corresponding to two choices of the pair $(\varepsilon_1, \varepsilon_2)$, then in their intersection the two curves \mathcal{C}_k above coincide, for the points of \mathcal{C}_k belong to the Fučík spectrum of (2)-(3), and the points in the intersection of the squares and below any one of the two curves \mathcal{C}_k do not belong to the Fučík spectrum of (2)-(3). \square

A similar proof shows the supremum in (19) is attained and there are no nontrivial solutions of (18) with $\bar{a} > \tilde{a}(r)$. This leads to

Theorem 3.5. *Fix $\lambda_k \neq 0$. If $\lambda_k < 0$ let $\tilde{R}_k =]-\infty, \lambda_{k+1}[^2$, and if $\lambda_k > 0$ let $\tilde{R}_k =]0, \lambda_{k+1}[^2$. There exists a continuous curve $\mathcal{D}_k \subset \tilde{R}_k$ through (λ_k, λ_k) (symmetric with respect to the line that bisects the odd quadrants and, if $k \neq 1$, the graph of a strictly decreasing function) such that $(a, b) \in \mathcal{D}_k$ implies (2)-(3) has a nontrivial weak solution. If (a, b) lies in \tilde{R}_k and above \mathcal{D}_k , then (2)-(3) has no nontrivial weak solution.*

Proof. The proof is similar to the one of Theorem 3.4 so we just point out the differences here. The image of the half lines $\bar{b} = r\bar{a}$, with \bar{a} increasing from zero to $+\infty$, are curves in the (a, b) -plane starting at $(\lambda_{k+1} - \varepsilon_3 - 1/\rho, \lambda_{k+1} - \varepsilon_3 - 1/\rho)$ and ending at $(\lambda_{k+1} - \varepsilon_3, \lambda_{k+1} - \varepsilon_3)$. Consider the curve \mathcal{D}_k parametrized by

$$r \mapsto \left(\lambda_{k+1} - \varepsilon_3 - \frac{1}{\tilde{a}(r) + \rho}, \lambda_{k+1} - \varepsilon_3 - \frac{1}{r\tilde{a}(r) + \rho} \right).$$

It passes through (λ_k, λ_k) and divides the square $R_k :=]\lambda_{k+1} - \varepsilon_3 - 1/\rho, \lambda_{k+1} - \varepsilon_3[^2$ in two connected components. We say that a point is inside R_k and *above* \mathcal{D}_k if it belongs to the component which contains $]\lambda_k, \lambda_{k+1} - \varepsilon_3[^2$. There are no points on the Fučík spectrum of (2)-(3) inside the square R_k and above \mathcal{D}_k . If $\lambda_k < 0$ then $R_k =]\lambda_{k+1} - \varepsilon_3 - 1/\varepsilon_4, \lambda_{k+1} - \varepsilon_3[^2$, and if $\lambda_k > 0$ then

$$R_k = \left] \varepsilon_4 \frac{(\lambda_{k+1} - \varepsilon_3)^2}{1 + \varepsilon_4(\lambda_{k+1} - \varepsilon_3)}, \lambda_{k+1} - \varepsilon_3 \right[.$$

We now write $R_k = R_k(\varepsilon_3, \varepsilon_4)$. If $\lambda_k < 0$ we define $\tilde{R}_k :=]-\infty, \lambda_{k+1}[^2$, and if $\lambda_k > 0$ we define $\tilde{R}_k :=]0, \lambda_{k+1}[^2$. If $(a, b) \in \tilde{R}_k$ then we can choose $\varepsilon_3, \varepsilon_4$ sufficiently small so that $(a, b) \in R_k(\varepsilon_3, \varepsilon_4)$. \square

4 An existence result for an asymptotically linear problem

We wish to prove the existence of a weak solution of

$$\begin{cases} \square u = au^+ - bu^- + p & \text{in }]0, \pi[\times \mathbb{T}, \\ u(0, t) = u(\pi, t) = 0 & \text{for } 0 \leq t \leq 2\pi. \end{cases} \quad (27)$$

As mentioned in the Introduction, our results should be compared with those of [3], [11], [15], [17], [24]. An important difference is that we are able to consider pairs (a, b) outside the square $[\lambda_{k-1}, \lambda_{k+1}]^2$.

Let us initially assume that (a, b) lies in \tilde{Q}_k and below \mathcal{C}_k . Furthermore we assume $p : [0, \pi] \times \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, with

$$\lim_{|u| \rightarrow +\infty} \frac{p(s, t, u)}{u} = 0 \quad \text{uniformly in } (s, t), \quad (28)$$

satisfying:

- (H1) There exists $\underline{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto au^+ - bu^- + p(s, t, u) - (\lambda_{k-1} + \underline{\varepsilon}_0)u$ is increasing.
- (H2) If $\lambda_k > 0$ there exists $\bar{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto p(s, t, u) - (1/\bar{\varepsilon}_0)u$ is decreasing.
- (H3) If $\lambda_k < 0$ there exists $\bar{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto au^+ - bu^- + p(s, t, u) + \bar{\varepsilon}_0 u$ is decreasing.

Choose ε_1 and ε_2 small enough so that $(a, b) \in Q_k(\varepsilon_1, \varepsilon_2) =]p_k, q_k[^2$, with Q_k given by (26) and μ given by (10). We recall that $p_k \searrow \lambda_{k-1}$ as $\varepsilon_1 \searrow 0$; if $\lambda_k > 0$ then $q_k \nearrow +\infty$ as $\varepsilon_2 \searrow 0$, and if $\lambda_k < 0$ then $q_k \nearrow 0$ as $\varepsilon_2 \searrow 0$. By decreasing ε_1 and ε_2 if necessary (so $p_k < \lambda_{k-1} + \underline{\varepsilon}_0$, and $q_k > (1/\bar{\varepsilon}_0) + \max\{a, b\}$ if $\lambda_k > 0$, $q_k > -\bar{\varepsilon}_0$ if $\lambda_k < 0$), we may assume that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the map

$$u \mapsto au^+ - bu^- + p(s, t, u) - p_k u \quad \text{is strictly increasing,} \quad (29)$$

and the map

$$u \mapsto au^+ - bu^- + p(s, t, u) - q_k u \quad \text{is strictly decreasing.} \quad (30)$$

Therefore, for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $p(s, t, \cdot)$ is absolutely continuous. In addition, it is true that $p_k < b + p'_u(s, t, \cdot) < q_k$ almost everywhere on \mathbb{R}^- , and $p_k < a + p'_u(s, t, \cdot) < q_k$ almost everywhere on \mathbb{R}^+ . Here $p'_u(s, t, \cdot)$ denotes the derivative with respect to the third variable. Conversely, if these inequalities for the derivatives p'_u hold for some p_k and q_k with $]p_k, q_k[^2 = Q_k$, then (H1)-(H3) hold. In other words,

Remark 4.1. (H1)-(H3) hold if and only if the derivative of the nonlinear term of the differential equation with respect to u is between p_k and q_k almost everywhere for some p_k and q_k with $]p_k, q_k[^2 = Q_k$.

Define $P, J : [0, \pi] \times \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ by $P(s, t, u) = \int_0^u p(s, t, \tau) d\tau$ and

$$J(s, t, u) = H(u) + P(s, t, u),$$

with H given by (6). From (29), for each (s, t) the function $J(s, t, \cdot)$ is convex. Denote by $J^*(s, t, \cdot)$ the Fenchel-Legendre transform of $J(s, t, \cdot)$:

$$J^*(s, t, v) = \sup_{u \in \mathbb{R}} [vu - J(s, t, u)]. \quad (31)$$

Let Q be such that

$$J^*(s, t, v) = H^*(v) + Q(s, t, v).$$

and $q := Q'_v$. Under our assumptions $Q(s, t, \cdot)$ is C^1 as $J(s, t, \cdot)$ is strictly convex and superlinear (see [14, Proposition 2.4]). Also, for any $v \in \mathbb{R}$, the map $(s, t) \mapsto J^*(s, t, v)$ is measurable since the supremum in (31) can be taken over the set of rationals. The function J^* is Carathéodory.

Lemma 4.2. Under assumption (28), we have

$$\lim_{|v| \rightarrow +\infty} \frac{q(s, t, v)}{v} = 0 \quad (32)$$

uniformly in (s, t) .

Proof. We denote by $\check{a} = a - \lambda_{k-1} - \varepsilon_1$ and $\check{b} = b - \lambda_{k-1} - \varepsilon_1$. Let

$$v = J'_u(s, t, u) = \check{a}u^+ - \check{b}u^- + p(s, t, u).$$

We see that $|u| \rightarrow +\infty$ is equivalent to $|v| \rightarrow +\infty$. Suppose $0 < \varepsilon < \min \left\{ \frac{1}{\check{a}}, \frac{1}{\check{b}} \right\}$. We choose c_1 large enough so that for all $|u| > c_1$ we have

$$\left| \frac{p(s, t, u)}{u} \right| < \frac{1}{2} \min\{\check{a}^2, \check{b}^2\} \varepsilon.$$

Take c_2 such that $|v| > c_2$ implies $|u| > c_1$. In the first place, assume $v > c_2$. Then

$$v = v^+ = \check{a}u^+ \left(1 + \frac{p(s, t, u)}{\check{a}u} \right),$$

or

$$u^+ = \frac{1}{\check{a}}v^+ \frac{1}{1 + p(s, t, u)/(\check{a}u)} = \frac{1}{\check{a}}v^+(1 + y),$$

where $|y| < \frac{2}{\check{a}} \left| \frac{p(s, t, u)}{u} \right| < \check{a}\varepsilon$. Indeed, we have used $1/(1 - x) = 1 + y$ with $|y| < 2|x|$ for $|x| < 1/2$. On the other hand,

$$u = (J^*)'_v(s, t, v) = \frac{1}{\check{a}}v^+ - \frac{1}{\hat{b}}v^- + q(s, t, v) = \frac{1}{\check{a}}v^+ \left(1 + \check{a} \frac{q(s, t, v)}{v} \right).$$

It follows that $y = \check{a} \frac{q(s, t, v)}{v}$ and so

$$\left| \frac{q(s, t, v)}{v} \right| < \varepsilon. \quad (33)$$

Similarly for $v < -c_2$. We have showed that $|v| > c_2$ implies (33). This proves the lemma. \square

Clearly (32) yields

$$\lim_{|v| \rightarrow +\infty} \frac{Q(s, t, v)}{v^2} = 0 \quad (34)$$

uniformly in (s, t) . Problem (27) is equivalent to

$$(L - \mu)v = (J^*)'_v(\cdot, \cdot, v) - \mu v = \hat{a}v^+ - \hat{b}v^- + q(\cdot, \cdot, v), \quad (35)$$

with v given by

$$v = J'_u(s, t, u).$$

This is the Euler-Lagrange equation for the functional $I : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$I(v) = \frac{1}{2} \langle (L - \mu)v, v \rangle - \int \int_{[0, \pi] \times \mathbb{T}} \left[J^*(s, t, v) - \frac{\mu}{2} |v|^2 \right] ds dt.$$

Proposition 4.3. *The functional I has an absolute maximum.*

Proof. Let $0 < c < \min\{a - p_k, b - p_k\}$. There exists $d \geq 0$ such that $J(s, t, u) \geq \frac{c}{2}u^2 - d$ as the function p is continuous. This leads to $J^*(s, t, v) \leq \frac{1}{2c}v^2 + d$. On the other hand $J(s, t, 0) = 0$ implies $J^*(s, t, v) \geq 0$. So since J^* is Carathéodory, the functional I is well defined.

By (29), the function $u \mapsto J(s, t, u)$ is twice differentiable almost everywhere and its second distributional derivative is nonnegative. Moreover, (30) implies that the function $u \mapsto \frac{a}{2}(u^+)^2 + \frac{b}{2}(u^-)^2 + P(s, t, u) - \frac{q_k}{2}u^2 = J(s, t, u) - \frac{1}{2\mu}u^2$ is twice differentiable almost everywhere and its second distributional derivative is nonpositive. We conclude that the second distributional derivative of $J(s, t, \cdot)$ coincides with its absolutely continuous part. Furthermore, $J''_u(s, t, u) \leq 1/\mu$ almost everywhere. On the other hand, since $J^*(s, t, \cdot)$ is convex, its second distributional derivative is nonnegative. The density of the absolutely continuous part of the second distributional derivative of $J^*(s, t, \cdot)$ at $J'_u(s, t, u)$ is $(J^*)''_v(J'_u(s, t, u))$ (derivative in the Alexandrov sense). At any point at which this derivative exists $(J^*)''_v(J'_u(s, t, u)) = [J''_u(s, t, u)]^{-1} \geq \mu$ (see [23, pp. 58-59]). Therefore the second distributional derivative of the map $v \mapsto J^*(s, t, v) - \frac{\mu}{2}|v|^2$ is nonnegative and this map is convex.

For $\check{a}(r)$ as in (22) and all $v \in \mathcal{H}$, we know

$$\langle (L - \mu)v, v \rangle - \check{a}(r)\|v^+\|^2 - r\check{a}(r)\|v^-\|^2 \leq 0.$$

Let (\hat{a}, \hat{b}) be the point given by (12). Then

$$\langle (L - \mu)v, v \rangle - \hat{a}\|v^+\|^2 - \hat{b}\|v^-\|^2 \leq -c\|v\|^2, \quad (36)$$

with $c = \min \left\{ \hat{a} - \check{a} \left(\frac{\hat{b}}{\hat{a}} \right), \hat{b} - \check{a} \left(\frac{\hat{a}}{\hat{b}} \right) \right\} > 0$.

Let v_n be a maximizing sequence. Using (34) and (36), we easily see that the sequence (v_n) is bounded in \mathcal{H} . We write $v_n = w_n + z_n$ with $w_n \in \mathcal{R}$ and $z_n \in \mathcal{N}$. Modulo a subsequence, we know $v_n \rightharpoonup v_0$, $w_n \rightharpoonup w_0$ and $z_n \rightharpoonup z_0$, all in \mathcal{H} . We prove that $\limsup \|w_n\| = \|w\|$ and $\limsup \|z_n\| = \|z\|$. So suppose that $\|w_n\|$ and $\|z_n\|$ converge. Since $Lw_n \rightarrow Lw_0$ strongly in \mathcal{H} , we have

$$\begin{aligned} \sup I &\leq I(v_0) - \mu \lim(\|w_n\|^2 - \|w_0\|^2) + \nu \lim(\|z_n\|^2 - \|z_0\|^2) \\ &\quad - \left(\liminf \iint_{[0, \pi] \times \mathbb{T}} \left[J^*(s, t, v_n) - \frac{\mu}{2}|v_n|^2 \right] ds dt \right. \\ &\quad \left. - \iint_{[0, \pi] \times \mathbb{T}} \left[J^*(s, t, v_0) - \frac{\mu}{2}|v_0|^2 \right] ds dt \right) \end{aligned}$$

The functional $v \mapsto \iint_{[0, \pi] \times \mathbb{T}} \left[J^*(s, t, v) - \frac{\mu}{2}|v|^2 \right] ds dt$ is weakly lower semi-continuous in \mathcal{H} because $v \mapsto J^*(s, t, v) - \frac{\mu}{2}|v|^2$ is convex and bounded below (using (34), as $\hat{a}, \hat{b} > 0$). We conclude that

$$I(v_0) = \sup I$$

and $v_n \rightarrow v_0$ in \mathcal{H} . □

Theorem 4.4. *Let $p : [0, \pi] \times \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (28) and (H1) to (H3). Suppose (a, b) lies in \tilde{Q}_k and below \mathcal{C}_k . Then problem (27) has a weak solution.*

Proof. Any maximum point v_0 of I is a solution of (35). The equivalence between (27) and (35) implies $u_0 = Lv_0$ is a weak solution of problem (27). \square

A similar result holds if (a, b) lies in \tilde{R}_k and above \mathcal{D}_k . In this case, (H1) to (H3) above should be replaced by

- (H1') There exists $\underline{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto au^+ - bu^- + p(s, t, u) - (\lambda_{k+1} - \underline{\varepsilon}_0)u$ is decreasing.
- (H2') If $\lambda_k > 0$ there exists $\bar{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto au^+ - bu^- + p(s, t, u) - \bar{\varepsilon}_0 u$ is increasing.
- (H3') If $\lambda_k < 0$ there exists $\bar{\varepsilon}_0 > 0$ such that for each $(s, t) \in [0, \pi] \times \mathbb{T}$ the function $u \mapsto p(s, t, u) + (1/\bar{\varepsilon}_0)u$ is increasing.

Theorem 4.5. *Let $p : [0, \pi] \times \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (28) and (H1') to (H3'). Suppose (a, b) lies in \tilde{R}_k and above \mathcal{D}_k . Then problem (27) has a weak solution.*

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